An overview of Coherent States

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Coherent states (including Glauber states) are elegant and very helpful for understanding what goes on in oscillating systems, including harmonic oscillators as well as many kinds of waves. They show us the correspondence between quantum oscillators and classical oscillators. Among other things, they make it clear that even in fully quantum mechanical systems, not everything is quantized, not all waves are quantized, not all states are discrete, et cetera. In this review article essential mathematical properties of coherent states have been explored.

Coherent States | Quantum Optics | Harmonic Oscillators

Coherent States(Introduction)

Coherent States- an exotic phenomenon that relates Quantum Optics to classical optics. In physics, in quantum mechanics, a coherent state is the specific quantum state of the quantum harmonic oscillator whose dynamics most closely resembles the oscillating behaviour of a classical harmonic oscillator. Coherent states play an important role in representing quantum dynamics, particularly when the quantum evolution is close to classical. The coherent state was originally introduced by Schrodinger in 1926 as a Gaussian wavepacket to describe the evolution of a harmonic oscillator. The centroid (mean values of the canonical variables) obtained from the Gaussian wavefunction follows the classical evolving harmonic oscillator, thereby mimicking its periodic evolution, and the spread of the wavepacket is fixed. Furthermore, the spread (variance) of this wavefunction satisfies the Heisenberg uncertainty relation and hence is as localized as possible within the constraints of quantum theory. In this brief review we attempt to give an overview of the physical realisation of coherent states and the mathematics governing it in different situations and try to correlate it with the classical trajectory. Fig 1.1 gives us a pictorial description of coherent states and it's position and momentum distribution.

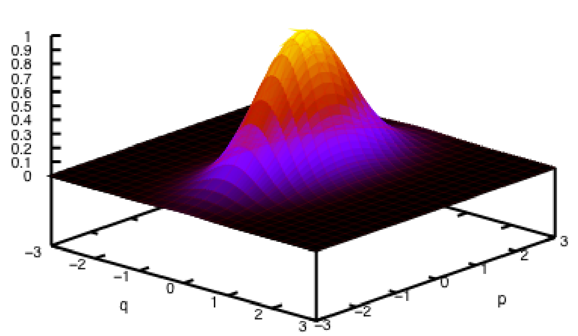

Fig. 1. Coherent States

Harmonic Oscillators

In classical mechanics we can talk about the position of a particle at any given time: x(t). The quantum mechanics analog to this is a particle's wavefunction:

$$|\psi(x,t)|\,.$$

This wavefunction has a statistical interpretation, $|\psi(x,t)|^2$ gives the probability of finding the particle at position x at time t. More precisely we could say that $\int_a^b |\psi(x,t)|^2\,dx$ is the probability of finding the particle between a and b, at time t. The wavefunction can be obtained by solving the Schrodinger equation. The time independent Schrodinger equation for a harmonic oscillator is given by:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}kx^2=E\Psi$$

The solution to this equation is given by the following wave functional form:

$$|\Psi_n\rangle=\sqrt[4]{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{1}{2^n n!}}e^{-y^2}H_n(y)$$

where:

$$y=\sqrt{\frac{m\omega}{\hbar}}x$$

and H is the Hermite polynomial of order n. Fig. 2.1 illustrates the idea of energy levels which are equispaced in a harmonic oscillator.

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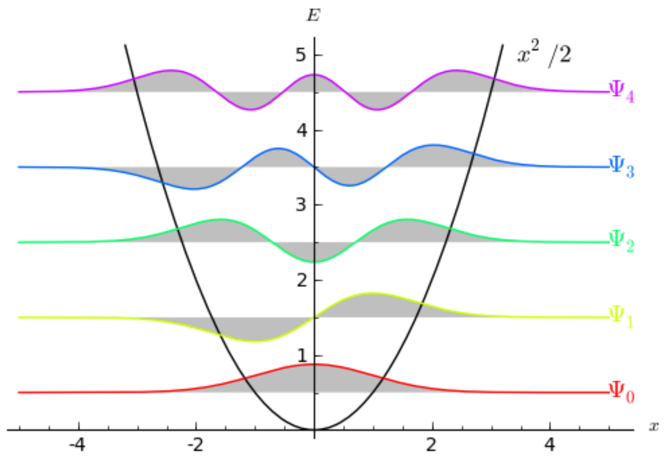

Fig. 2. Coherent States

Coherent States for Harmonic Oscillator

We know that the coherent states are minimum uncertainty states for a given system. Hence we expect classical behaviour from the expectation values of position and momentum for the coherent states. We mention the Heisenberg's uncertainty principle briefly here :

$(\sigma_x)(\sigma_p) \geq \frac{1}{2i} \langle [x,p] \rangle$

Wavefunctions that satisfy the Heisenberg uncertainty principle with equality are called minimum uncertainty wavefunctions. Coherent states are seen to satisfy the above criterion. Before starting with the proofs it is best to introduce the concept of ladder operators. a is called the annihilation/destruction/lowering operator and is given by:

$a = (m\omega x + ip)\frac{1}{\sqrt{2m\omega\hbar}}$

,whereas $a\dagger$ is called the creation/raising operator and is given by:

$a\dagger = (m\omega x - ip)\frac{1}{\sqrt{2m\omega\hbar}}$

These operators satisy the following relation :
$[a, a\dagger] = 1$
and :
$a \dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$
$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$

Moreover, the ground state wavefunction $|0\rangle$ of the harmonic oscillator is a minimum uncertainty wavefunction, whereas the wavefunctions $|n\rangle$ are not minimum uncertainty wavefunctions. The following has been proved :

Proof 1: Ground State wave functions are minimum uncertainty states.

$$\langle 0|(a + a\dagger)(a + a\dagger)|0\rangle = \langle 0|(aa + aa\dagger + a\dagger a + a\dagger a\dagger)|0\rangle \quad [1]$$

Since $a\,|0\rangle = 0$ and $\langle 0|2\rangle = 0$,
We have the required expression =

$\langle 0 \,|(\frac{H}{\omega\hbar})|\, 0\rangle = \frac{E_0}{\omega\hbar} + \frac{1}{2} = 1$

Similarly,

$\langle 0|(a - a\dagger)(a - a\dagger)|0\rangle = -1$

Also, the following relations must be noted:

$\langle x\rangle_0 = 0$

$\langle p\rangle_0 = 0$

$\langle x^2\rangle_0 = \frac{\hbar}{2m\omega}$

$\langle p^2\rangle_0 = \frac{m\omega\hbar}{2}$

Now when we calculate the uncertainties in x and p we get the following:

$(\sigma_x)_0^2 = \langle x^2\rangle_0 - \langle x\rangle_0^2$

$(\sigma_p)_0^2 = \langle p^2\rangle_0 - \langle p\rangle_0^2$

Finally, $(\sigma_x)_0^2(\sigma_p)_0^2 = \frac{\hbar^2}{4}$ (minimum uncertainty relation)

We now attemp to make another proof.

Proof 2: Any state $|n\rangle$ is not a minimum uncertainty state. Since wavefunctions are orthonormal we have ,

$\langle x\rangle_n = 0$

$\langle p\rangle_n = 0$

$\langle x^2\rangle_n = \frac{\hbar}{2m\omega}(1+2n)$

$\langle p^2\rangle_n = \frac{m\omega\hbar}{2}(1+2n)$

Yielding the uncertainty relation as ,

$(\sigma_x)_0^2(\sigma_p)_0^2 = \frac{\hbar^2}{4}(1+2n)^2$

The reason behind the fact that any arbitray $|n|\rangle$ state does not statisfy the coherent states condition is that $a\,|n\rangle \neq 0$.

Finally, we show that Coherent States satisfy the minimum uncertainty relation. We now introduce a new definition for the coherent states:
The states $|\alpha\rangle$ defined by:

$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$

, with $\langle\alpha|\alpha\rangle = 1$, are called coherent states.

Proof 3: Coherent States satisfy the minimum uncertainty relation.

$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$,

$\langle\alpha|\,a\dagger = \langle\alpha|\,\bar{\alpha}$

$\langle\alpha\,|a\dagger a|\rangle = |\alpha|^2$

Also,
$\langle \alpha \, | a + a\dagger | \rangle = \alpha + \bar{\alpha}$

$\langle \alpha \, | a - a\dagger | \rangle = \alpha - \bar{\alpha}$

$\langle \alpha \, | (a + a\dagger)^2 | \rangle = (\alpha + \bar{\alpha})^2 + 1$

$\langle \alpha \, | (a - a\dagger)^2 | \rangle = (\alpha - \bar{\alpha})^2 + 1$

Now when we calculate the uncertainties in position and momentum, we get:

$(\sigma_x)_0^2 = \frac{\hbar}{2m\omega}$

$\langle p^2 \rangle_n = \frac{m\omega\hbar}{2}$

yielding,

$(\sigma_x)_0^2(\sigma_p)_0^2 = \frac{\hbar^2}{4}$ (which shows that coherent states are minimum uncertainty states).

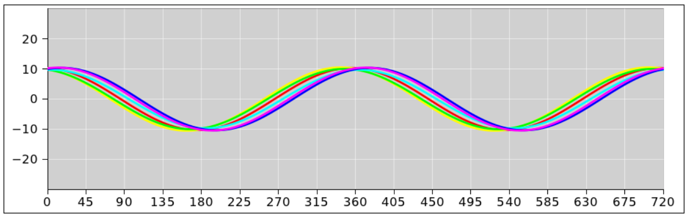

Fig. 3. x-waveform of coherent state

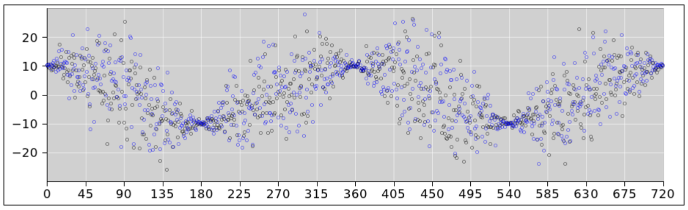

Fig. 4. Scatter Plot for coherent State

N-Space representation. Using the above definition and some operators we will derive the expression for Coherent States in the n-space representation.

$|\alpha\rangle = \sum_{n=0}^{\infty} c_n \, |n\rangle$,where $c_n \epsilon R$.

$\langle m|\alpha\rangle = \sum_{n=0}^{\infty} c_n \, \langle m|n\rangle = c_m$

$|\alpha\rangle = \sum_{n=0}^{\infty} \langle n\rangle \, \langle n|\alpha\rangle$

Now, we the following relation, $|n\rangle = \frac{(\hat{a\dagger})^n}{\sqrt{n!}} \, |0\rangle$ and get ,

$\langle n|\alpha\rangle = \frac{1}{\sqrt{n!}} \, \langle 0|(\hat{a})^n|\alpha\rangle$

$= \frac{\alpha^n}{\sqrt{n!}} \, \langle 0|\alpha\rangle$

$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, \langle 0|\alpha\rangle \, |n\rangle$

$= e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |n\rangle$

We can also show that this coherent state is not an eigenstate solution for the Harmonic Oscillator Schrodinger Equation:

$H \, |\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} E_n \, |n\rangle \neq \lambda \, |\alpha\rangle$

These coherent states can also be thought to be displaced ground states of the Harmonic Oscillator using the Displacement operator D:

$|\alpha\rangle = D(\alpha) \, |0\rangle$,where

$D(\alpha) = e^{\alpha\hat{a\dagger} - \bar{\alpha}\hat{a}}$.

We check the time evolution of the coherent states to find that as these states evolve with time they remain coherent.

Time evolution. The time evolution of a state is given by the time evolution operator **U**(t). Using what we know about this operator and what we have derived in the previous sections we can deduce:

$|\alpha, t\rangle = U(t, 0) \, |\alpha(0)\rangle = e^{\frac{-iHt}{\hbar}} e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} E_n \, |n\rangle \neq \lambda \, |\alpha\rangle$

But the $|n\rangle$:s are eigenstates of the hamiltonian so:

$|\alpha, t\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} E_n \, |n\rangle \neq \lambda \, |\alpha\rangle \, e^{\frac{-i(n+\frac{1}{2})\hbar t}{\hbar}}$

which is the same as :

$|\alpha, t\rangle = e^{\frac{-|\alpha|^2}{2}} e^{\frac{-it\omega}{2}} \sum_{n=0}^{\infty} \frac{(\alpha(0)e^{-it\omega}a\dagger)^n}{n!} \, |0\rangle =$

$exp(-\frac{1}{2}|\alpha|^2 - \frac{i}{2}t\omega + \alpha(0)e^{-it\omega}a\dagger) \, |0\rangle$

The first and the third time in the exponent, operating on the ground state will give us a coherent state with the time dependent eigenvalue $e^{-it\omega}\alpha(0)$ while the second term only will contribute with a phase factor. Thus we have:

$|\alpha, t\rangle = e^{-\frac{it\omega}{2}} \, |e^{-it\omega}\alpha(0)\rangle = |\alpha(t)\rangle$

So the coherent state remains coherent under time evolution.

In summary, we have seen that the coherent states are minimal uncertainty wavepackets which remains minimal under time evolution. Furthermore, the time dependant expectation values of x and p satifies the classical equations of motion. From this point of view, the coherent states are very natural for studying the classical limit of quantum mechanics. Figure 3.1 shows the evolution of coherent states with time.

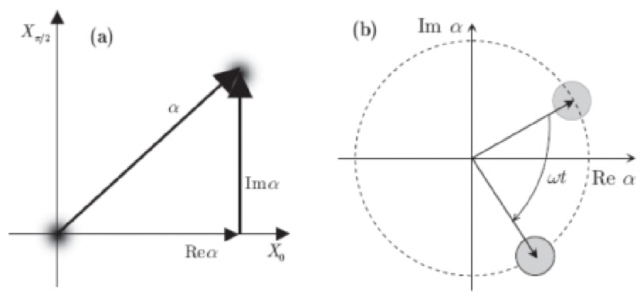

Fig. 5. Time Evolution

X-representation of Coherent States

We know that $\left\langle x^{'}|0\right\rangle$ is a minimal gaussian wave packet with $\langle x\rangle = \langle p\rangle = 0$. Since $\left\langle x^{'}|0\right\rangle$ is also a minimal wave packet and ,

$R(\alpha) = \left\langle\alpha|\frac{a+a\dagger}{2}|\alpha\right\rangle = (\frac{m\omega}{2\hbar})^{\frac{1}{2}}\left\langle\alpha|x|\alpha\right\rangle$

$Im(\alpha) = \left\langle\alpha|\frac{a-a\dagger}{2i}|\alpha\right\rangle = (\frac{1}{2m\omega\hbar})^{\frac{1}{2}}\left\langle\alpha|p|\alpha\right\rangle$

or,

$\langle x\rangle_\alpha = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}R(\alpha)$
$\langle p\rangle_\alpha = \sqrt{\hbar m\omega}\sqrt{2}Im(\alpha)$

It is usual to expect that $\left\langle x^{'}|0\right\rangle$ is a displaced gaussian moving with velocity $v = \frac{p}{m}$.
The space representation of the coherent states is found out to be :

$\psi_\alpha(x^{'}) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}e^{\frac{i}{\hbar}\langle p\rangle_\alpha x^{'} - \frac{m\omega}{2\hbar}(x^{'}-\langle x\rangle_\alpha)^2}$

Squeezing of Coherent States

Squeezed states are a general class of minimum-uncertainty states. Let us consider a hermitian operator A with variance $(\Delta A)^2 = \left\langle A^2\right\rangle - \langle A\rangle^2$, For any two hermitian operators,the uncertainty relation holds. We know that for the Harmonic Oscillator problem we define the hermitian field quadrature operators.

$X_1 = a + a\dagger$
$X_2 = -i(a - a\dagger)$
$=> [X_1, X_2] = 2i$

with the back transformation $a = \frac{X_1+iX_2}{2}$ leading to a simple statement of the uncertainty relation

$\Delta X_1\Delta X_2 \geq 1$

For a coherent state $|\alpha\rangle$, we find the complex number 2α as a cente of the error cycle in the complex $X_1 - X_2-$ plane :

$\langle\alpha|X_1|\alpha\rangle = \langle\alpha|a + a\dagger|\alpha\rangle = \alpha + \alpha^* = 2Re(\alpha)$
$\langle\alpha|X_2|\alpha\rangle = -i\langle\alpha|a - a\dagger|\alpha\rangle = -i(\alpha - \alpha^*) = 2Im(\alpha)$

and ,

$(\Delta X_1)^2 = \left\langle\alpha|X_1^2|\alpha\right\rangle - \langle\alpha|X_1|\alpha\rangle^2$
$= \left\langle\alpha|a^2 + aa\dagger + a\dagger a + a\dagger^2|\alpha\right\rangle - \langle\alpha|a + a\dagger|\alpha\rangle^2$

acting to the right with a and to the left with a†, we find:

$(\Delta X_1)^2 = (\alpha^2 + 1 + 2|\alpha|^2 + \alpha^{*2} - (\alpha + \alpha^*)^2) = 1$

The free time evolution of a coherent state under $H = \hbar\omega a\dagger a$ is given as a rotation of the error circle or ellipse around the origin.
In analogy to the coherent states, we define the squeeze operator,

$S(\epsilon) \equiv exp[\frac{\epsilon^*}{2}a^2 - \frac{\epsilon}{2}a\dagger^2]$
where, $\epsilon = re^{2i\phi}$

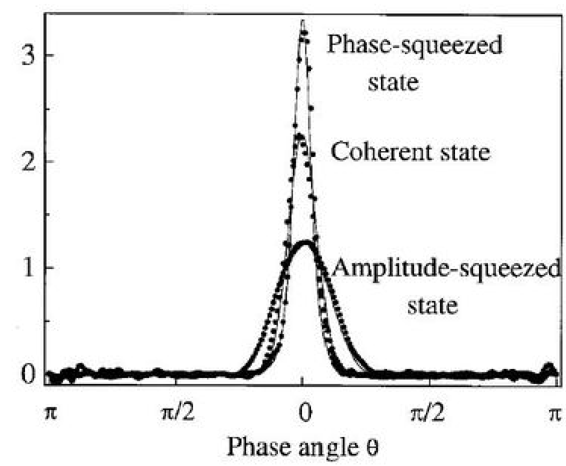

Fig. 6. Subpoissonian statistics

The squeeze operator obeys the relation,

$S\dagger(\epsilon) = S\dagger^{-1}(\epsilon) = S\dagger(-\epsilon)$

and this is unitary. It can be easily shown that the following results hold :-

$S\dagger(\epsilon)aS(\epsilon) = acosh(r) - a\dagger e^{-2i\phi}sinh(r)$
$S\dagger(\epsilon)a\dagger S(\epsilon) = a\dagger cosh(r) - ae^{-2i\phi}sinh(r)$
$-> S\dagger(\epsilon)(Y_1 + iY_2)S(\epsilon) = e^{-r}Y_1 + iY_2e^r$

where $Y_1 + iY_2 \equiv (X_1 + iX_2)e^{-i\phi}$ is a rotated complex amplitude. Thus the squeeze operator attenuates one component of the (rotated) complex amplitude while amplifying the other one. The degree of scaling is given by $r = |\epsilon|$ (squeeze factor). This will be elucidated by the definition of the squeezed state, first squeezing the vacuum (transform the error cycle at zero ellipse with the principle axis Y_1 and Y_2) and then displacing it (move it to 2α):

$|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)|0\rangle$

A squeezed state has the following expectation values and variances:

$\langle X_1 + iX_2 \rangle = \langle 2\alpha \rangle = \langle S\dagger(\epsilon) D\dagger(\alpha) a D(\alpha) S(\epsilon) \rangle$
$= 2\langle 0|S\dagger(\epsilon)(a+\alpha)S(\epsilon)|0\rangle = 2\alpha$

Thus the expectation value of $|\alpha, \epsilon\rangle$ state in the $X_1 - iX_2$ plane is the same as for $|\alpha\rangle$ state. Similarly we have

$\Delta Y_1 = e^{-r} \Delta Y_2 = e^{r}$
thus the state is squeezed in one direction and expanded in the other. The photon number mean and variance is :

$$\langle N \rangle = |\alpha|^2 + sinh^2(r)$$

$$(\Delta N)^2 = \left|\alpha cosh(r) - \alpha^* e^{2i\phi} sinh(r)\right|^2 + 2cosh^2(r) sinh^2(r)$$

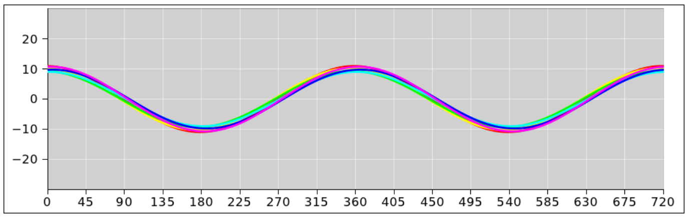

Fig. 7. x-waveform of squeezed coherent state

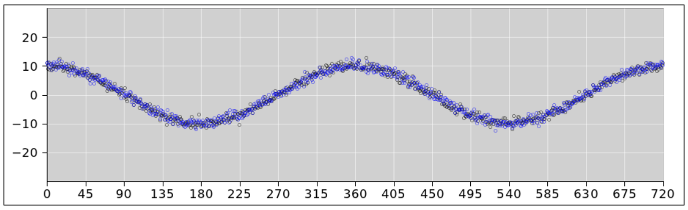

Fig. 8. Scatter Plot for squeezed coherent state

We can generate squeezed states by an Interaction Hamiltonian of the form,

$$H_I = \frac{\hbar[\chi a\dagger^2 + \chi^* a^2]}{2}$$

which describes simultaneous two-photon generation or absorption processes, as can be realised by second order non linear processed where a photon of energy $2\hbar\omega$ generates two photons, each of energy $\hbar\omega$.

Perturbation of Coherent States.

When we have a quantum system for which coherent states exist we can generate a perturbation in the system or a linear/non-linear function of space to study the effect of such disturbances on the system properties. For eg. if we see the Harmonic Oscillator the system has equispaced energy levels given by $E_n = (\hbar + \frac{1}{2})\omega$. We previously saw that we have coherent states given by $|\alpha\rangle$. If we add a perturbation say for eg. an uniform Electric field the changes are pretty interesting. We will first attempt to study the effect on the Energy and classical trajectory using algebraic approach. The equation for the Hamiltonian for a system of Harmonic oscillator with a particle of mass m,charge e and Electric field E_0 can be written as follows :

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - eE_0 x$$

Completing the square we get,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 (x - \frac{eE_0}{2m\omega^2})^2 - \frac{e^2 E_0^2}{2m\omega^2}$$

Writing $X = x - \frac{eE_0}{2m\omega^2}$ we get,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 X^2 + constant$$

The eigenvalue equation satisfied is,

$$H_0 |\psi\rangle = E_n |\psi\rangle$$

So by analogy for the perturbed system we have,

$$H |\psi\rangle = (H_0 + constant) |\psi\rangle = (E_n + constant) |\psi\rangle$$

So we can easily deduce that the energy eigenspectrum is shifted by a constant value which is proportional to the value of the Electric field (in general the strength of the perturbation) applied and the position is shifted to a value which is non-zero. Hence if we try to find the classical trajectory by evaluating the expectation values we should get :

$$\langle x \rangle = \langle x|\psi|x\rangle = \langle x\rangle_0 - \frac{eE_0}{2m\omega^2}$$

where $\langle x\rangle_0$ denotes the expectation value without the perturbation. However the expectation value of the momentum shall not change. We can solve this by using the linear perturbation theory of Quantum mechanics and we see that we arrive at the same result.
To see the effect on Coherent Startes we can instead of solving the equation use the x-t representation of the coherent states and replace the x term with the new x (X).
We recall that,

$$\psi_\alpha(x') = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{\frac{i}{\hbar}\langle p\rangle_\alpha x' - \frac{m\omega}{2\hbar}(x' - \langle x\rangle_\alpha)^2}$$

Here we know that the expectation value of the momentum will not change by the previous analogy, but the expectation value of x does change by a constant to give the representation as :

$$\psi_\alpha(x') = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{\frac{i}{\hbar}\langle p\rangle_\alpha x' - \frac{m\omega}{2\hbar}(x' - \left\langle (x - \frac{eE_0}{2m\omega^2})\right\rangle_\alpha)^2}$$

The added term on the exponential can go off as a constant phase and we get back the same expression essentially for the coherent states. So the coherent states in this case remain the same as far as mimicking the classical trajectories are concerned. The expectation values are shifted though but the nature of motion isn't. That is what we expect from classical theory too that in presence of a constant Electric field we expect the motion to remain unchanged. Hence the coherent states are important in a way that they bear information about the classical trajectory of the system and give us an idea of the classical behaviour of the system.

Generalised Coherent States

We can have an expression for generalised coherent states as a solution to the Schrodinger equation for a Harmonic Oscillator as follows:

$$\psi_{n,\alpha}(x,t) = \frac{(\omega/\pi)^{1/4}}{2^{n/2}\sqrt{n!}} e^{-\omega\frac{(x-\langle\hat{x}\rangle)^2}{2}} H_n[\sqrt{\omega}(x - \langle\hat{x}\rangle)]$$

$$* \, exp(i[-(n+1/2)\omega t + x\langle\hat{p}\rangle - \langle\hat{x}\rangle\langle\hat{p}\rangle/2])$$

where $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are the time-dependent expectation values of position and momentum in this state $\psi_{n,\alpha}(x,t) = \langle|n,\alpha\rangle$:

$$\langle\hat{x}\rangle = \langle n,\alpha|\hat{x}|n,\alpha\rangle = \sqrt{\frac{2}{\omega}}|\alpha|cos(\omega t - \theta),$$

$$\langle\hat{p}\rangle = \langle n,\alpha|\hat{p}|n,\alpha\rangle = -\sqrt{2\omega}|\alpha|sin(\omega t - \theta),$$

Hence we see that the this can serve as a generalised expression for coherent states. Of course by setting n=0, we get our usual coherent state which mimics the classical trajectory as seen earlier.

Importance of Coherent States/ Conclusion

The quantum mechanics tells us that if we increase the magnitude of the oscillation by a large factor, and zoom out accordingly, what we see is a particle oscillating with the expected frequency at the expected amplitude. In relative terms, all the probability is highly concentrated at the expected location. The absolute width of the distribution stays the same, but it relatively small compared to the large amplitude. However the following contrast is to be noted: Coherent States on one hand uphold the laws of quantum mechanics but neither are discrete nor are quantized. One can construct a coherent state anywhere you like, anywhere in phase space. Pick any real number x and any real number p, and then create a coherent state that is centered at the point (x, p) at time t=0. In some respects, the coherent state has properties which are relatively close to a classical state of the light field. For example, it resembles a classical oscillation of the light field, apart from some superimposed quantum noise, which is relatively weak for large average photon numbers. The quantum noise of the quadrature components of a Glauber state is equal. A nonlinear interaction can transform the circular uncertainty area into a deformed area with lower noise in one quadrature component; such states are called squeezed states of light. When such light fields experience linear losses, they are again pulled toward a coherent state. Hence we can see that coherent states form an integral part of the bridge between classical and quantum optics and it's study is essential to relate the classical properties with the quantum behaviour of the systems.

ACKNOWLEDGMENTS. This work was supported by Prof. Sayan Kar of Department of Physics, Indian Institute of Technology, Kharagpur.

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